Exam. Code : 211004
Subject Code : 4989

M.Sc. (Mathematics) 4th Semester

FUNCTIONAL ANALYSIS–II

Paper—MATH-581

Time Allowed—2 Hours] [Maximum Marks—100

Note :— There are **EIGHT** questions of equal marks. Candidates are required to attempt any **FOUR** questions.

1. (a) Define weak convergence and strong convergence of a sequence $\{x_n\}$ in a normed linear space X. Prove that every strongly convergent sequence is weakly convergent and give an example to show that the converse may not be true.

   (b) If $\{x_n\}$ is a weakly convergent sequence in a normed space X, which converges weakly to $x_0 \in X$ show that there is a sequence $\{y_m\}$ of linear combinations of elements of $\{x_n\}$ which converges strongly to x_0.

2. (a) Define adjoint of a linear operator on an inner product space. Prove that any bounded linear operator over a Hilbert space has an adjoint.

   (b) Let H be a Hilbert space over $\mathbb{C}$ and T be a bounded linear operator on H. Prove that T is normal if and only if $\|T^*(x)\| = \|T(x)\|$ for all $x \in H$.

3. (a) If P is a perpendicular projection on a Hilbert space H, prove that P is a positive operator such that $0 \leq P \leq I$.

   (b) Prove that any self-adjoint operator on a finite dimensional Hilbert space H has an eigenvalue.

4. State and prove Spectral theorem for normal operators on finite dimensional Hilbert space.

5. (a) Define compact linear map between two normed spaces. If X and Y are normed spaces and $F : X \to Y$ is a linear map, prove that F is a compact map if and only if for every bounded sequence $\{x_n\}$ in X, $\{F(x_n)\}$ has a subsequence which converges in Y.

   (b) Let X be a normed linear space and Y be a Banach space. Prove that the set of all compact linear operators from X and Y is a closed subspace of the space of all bounded linear operators from X to Y.

6. (a) If X is a normed linear space and T is compact linear operator on X. Prove that every nonzero spectral value of T is an eigen value of T.

   (b) If T is a compact linear operator on a normed space X. Prove that every eigenspace of T corresponding to a non-zero eigenvalue of T is finite dimensional.

7. (a) Define a Banch algebra. If G denote the set of all regular elements of a complex Banach algebra A, prove that the mapping $x \to x^{-1}$ is a homeomorphism of G onto itself.

   (b) Prove that the mapping $\lambda 1 \to \lambda$ is an isometric isomorphism of a complex Banach algebra A to $\mathbb{C}$ if and only if 0 is the only topological divisor of zero in A.

8. (a) Define spectral radius of an element x of a complex Banach algebra A. Prove that spectral radius of any element x of a complex Banach algebra A equals $\lim \| x^n \|^{1/n}$.

   (b) Let A be a complex Banach algebra, S and Z denote the set of singular elements of A and set of topological divisors of zero in A respectively. Prove that Z is a subset of S and boundary of S is a subset of Z.